THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Problem Set 5: Continuous Functions

1. Use the $\epsilon - \delta$ definition, show that $f(x) = x^3$, $x \in \mathbb{R}$ is a continuous function.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$

   Use the $\epsilon - \delta$ definition to show that f is continuous at 0.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

   Show that f is continuous at 0.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

   Show that f is discontinuous everywhere.

5. Prove that any polynomial is a continuous function on $\mathbb{R}$.

6. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.

7. (a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for all $r \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) Suppose that $g, h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $g(r) = h(r)$ for all $r \in \mathbb{Q}$. Prove that $g(x) = h(x)$ for all $x \in \mathbb{R}$.

8. (**Thomas's function**) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x \in \mathbb{Q} \text{ in lowest terms and } q > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that f is continuous at every irrational number but discontinuous at every rational number.

9. Let $A \subseteq \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be a function that is continuous at $c \in (a, b)$. Suppose that $f(c) > 0$, prove that there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a periodic function with period $T > 0$, i.e. T is the less positive number such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. Suppose that the function is continuous at every point $x \in [0, T]$, show that f is continuous everywhere.

11. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be additive if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that if f is continuous at some point x_0, then it is continuous at every point of $\mathbb{R}$.

12. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in [a, b]$. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in [a, b]$.

13. Show that every polynomial of odd degree with real coefficients has at least one real root.

14. Let f be continuous function on the interval $[0, 1]$ to $\mathbb{R}$ and such that $f(0) = f(1)$. (Remark: Therefore, f can be regarded as a continuous function on a circle.) Prove that there exists $c \in [0, 1/2]$ such that $f(c) = f(c + 1/2)$. (Hint: Consider $g(x) = f(x) - f(x - 1/2)$.) Conclude that there are, at any time, antipodal points on the earth's equator that have the same temperature.

15. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. Let $h : [a, b] \to \mathbb{R}$ be a function defined by

$$h(x) = \max\{f(x), g(x)\} \qquad \text{for } x \in [a, b].$$

Is $h(x)$ a continuous function? Why?

16. Let the function $f : \mathbb{R} \to \mathbb{R}$ be continuous and suppose that its image $f(\mathbb{R})$ is bounded.

    (a) Prove that there is a solution of the equation $f(x) = x$.

    (b) Furthermore, suppose f is strictly increasing on $\mathbb{R}$ and $a \in \mathbb{R}$ such that $f(a) > a$. Let $\{a_n\}$ be a sequence defined by $a_1 = a$ and $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$. Show that $\{a_n\}$ converges to a solution of the equation in (a).

    (c) Hence, find an approximate solution of the equation $\dfrac{e^x}{1 + e^x} = x$.

17. Let K be a compact nonempty subset of $\mathbb{R}$ and suppose that the function $f : K \to \mathbb{R}$ is continuous. Prove that $f(K)$ is also a compact set in $\mathbb{R}$.

18. (Generalized Max-Min Theorem) Let K be a compact nonempty subset of $\mathbb{R}$ and suppose that the function $f : K \to \mathbb{R}$ is continuous. Then f attains both an absolute maximum and an absolute minimum.

19. Prove that the function $f : [2, +\infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{x - 1}$ is uniformly continuous.

20. Prove that the function $f : [0, +\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x + 1} - \sqrt{x}$ is uniformly continuous.

21. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

22. Suppose that $f : (a, b) \to \mathbb{R}$ is continuous and monotone. Prove that $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if its image $f(a, b)$ is bounded.

23. If f is uniformly continuous on $A \subseteq \mathbb{R}$, and $|f(x)| \geq k \geq 0$ for all $x \in A$, show that $1/f$ is uniformly continuous on A.

24. If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if $\{x_n\}$ is a Cauchy sequence in A, the $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Furthermore, if f is continuous on A only, is the same conclusion true?